169489

OFFICE OF NAVAL RESEARCH

Contract N7onr-3:801

T. O. I.

NR-041-032

Technical Report No. 94

ON THE VELOCITY DISTRIBUTION OF TURBULENT FLOW BEHIND A SYSTEM OF THIN CYLINDRICAL RODS

by

R. Gran Olsson

GRADUATE DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

PROVIDENCE, R. I.

August, 1953

A11-94 10

On the velocity distribution of turbulent flow behind a system of thin cylindrical rods

By R. Gran Olsson*

1. Introduction. In an earlier paper the author has treated the problem of the velocity distribution of turbulent flow behind a system of rods spaced at equal distances from each other [1]. For the shear stress of turbulent flow the expression in terms of the mixing length due to L. Prandtl was assumed, where the mixing length at a certain distance from the rods may be put equal to a constant l_0 [2]. The velocity distribution was given by a non-linear differential equation of the second order, the solution of which can be expressed by quadrature. The purpose of this note is to show, how a solution of the differential equation can also be expressed in terms of the elliptic functions of Weierstrass [3].

2. The differential equation. We consider a system of cylindrical equidistant rods of infinite length in a plane perpendicular to the direction of flow in a stream of originally constant velocity. The problem is to calculate the distribution of velocity in the region behind the rods by means of the expression of L. Prandtl for the shear stress in turbulent flow. The mixing length will also be constant in a plane parallel to the plane of the rods because of their uniform spacing.

The coordinate system may be chosen as follows: The y-axis parallel to the plane of the rods and perpendicular

* Visiting Professor of Applied Mathematics at Brown University, Providence, Rhode Island.

to their axes, the x-axis through the axis of one of the rods and parallel to the direction of the undisturbed flow (Fig. 1). The components of the velocity may be noted by u and v, the velocity of the undisturbed flow by U. Behind the rod system at a sufficiently large distance the components of velocity are

$$u = U - u_1, \qquad v = v_1, \tag{1}$$

where u_1 and v_1 are assumed to be small compared with U. In establishing the equations of motion all terms of higher order of magnitude are dropped; furthermore the pressure term is neglected. This assumption can be justified by means of the solution obtained.

The equation of motion in the x-direction is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial \tau}{\partial y} \, . \tag{2}$$

Introducing $u = U - u_1$, $v = v_1$, where $u_1 << U$, $v_1 << U$, we obtain as a first approximation the equation

$$-U \frac{\partial u_1}{\partial x} = \frac{1}{\rho} \frac{\partial \tau}{\partial y} \tag{2a}$$

Because of the periodicity of the rod system, it seems convenient to assume the following expression for u_1:

$$u_1 = Af(x)\cos \alpha y, \tag{3}$$

where α has to be chosen in such a manner that the period of u_1 agrees with the distance λ of the rods, i.e. $\alpha = 2\pi/\lambda$.

The expression for shear stress according to the theory of mixing length is

$$\tau = \rho \ell^2 \left|\frac{\partial u}{\partial y}\right| \frac{\partial u}{\partial y} \tag{4}$$

The differential equation (2a) will then be transformed into

$$\frac{\partial u_1}{\partial x} = -k \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial y^2} \tag{2b}$$

where $k = 2\ell^2/U$ is a constant parameter.

3. The solution of the differential equation. If we make the substitution

$$u_1 = A_o x^{-1} F_1(y) \tag{5}$$

with the condition $F_1(0) = 1$, A_o may be interpreted as the amplitude of the velocity distribution, and we get the following differential equation

$$F_1(y) = A_o k F_1'(y) F_1''(y) \tag{2c}$$

Assume now $y = c\eta$, where c is a constant of the dimension length, we obtain the following equation

$$F_1(c\eta) = \frac{A_o k}{c^2} F_1'(c\eta) F_1''(c\eta) \tag{2d}$$

where on the right side the differentiations are made with respect to the variable η. As c is arbitrary we choose it so that

$$A_o k/c^2 = 1$$

and obtain the differential equation

$$F_1 = F_1' \cdot F_1'' \tag{2e}$$

which, multiplied by F_1', results in

$$F_1 F_1' = F_1'^2 F_1'' \tag{2f}$$

and once integrated

$$\frac{1}{2} F_1^2 = \frac{1}{3} F_1'^3 + C, \qquad (2g)$$

where C is a constant of integration, which can be determined by the condition that for $y = 0$ ($\eta=0$) $F_1 = 1$, and $F_1' = 0$ or $C = 1/2$.

With this value of C we get

$$F_1'^3 = -\frac{3}{2}(1 - F_1^2) \qquad (2h)$$

From the substitution

$$F_1^2 = 4z^3 + 1, \qquad (6)$$

$$dF_1 = 6z^2\, dz/F_1 = 6z^2\, dz/(1 + 4z^3)^{1/2} \qquad (6a)$$

the independent variable η is given by

$$\eta = (6)^{2/3} \int_z^0 \frac{z\, dz}{\sqrt{4z^3+1}}, \qquad (7)$$

which is an elliptic integral of Weierstrass type. To get a more convenient expression for η and F, we introduce the parameter w by means of the elliptic integral

$$w = \int_z^{\infty} \frac{dz}{(4z^3-g_2z-g_3)^{1/2}}, \qquad (8)$$

where the inverse function

$$z = p(w) \qquad (9)$$

is the elliptic p-function of Weierstrass. By comparison of the integrals (7) and (8), we see that in our case $g_2 = 0$, $g_3 = -1$. From (8) and (9) one obtains

$$\eta = (6)^{2/3} \int_w^{w_o} p(w)dw = (6)^{2/3} [\rho(w) - \rho(w_o)] \qquad (7a)$$

where w_0 corresponds to $z = 0$, u to an arbitrary value of z and $\rho(u)$ to the elliptic ρ-function of Weierstrass defined by the integral

$$\rho(w_0) = -\int p(w)dw$$

The function $F_1(\eta)$ of velocity distribution is given by Eqs. (6) and (9)

$$F_1(w) = \left\{4p^3(w) + 1\right\}^{1/2} \qquad (6a)$$

On the other hand the derivative of the Weierstrass p-function is given by

$$p'(w) = \left\{4p^3(w) - g_2 p(w) - g_3\right\}^{1/2} \qquad (10)$$

so that the velocity distribution is given by

$$F_1(w) = p'(w) \qquad (6b)$$

Thus we have a representation of the variables F_1 and η by means of the common variable u. The result of the numerical calculation is shown in Fig. 2 and is obtained by using the tables of the Jacobian elliptic functions by L. M. Milne-Thomson [4]. The p - function of Weierstrass can be expressed by the Jacobian elliptic function in the following manner

$$p(w) = e_2 + H\,\frac{1+cn(2wH^{1/2})}{1-cn(2wH^{1/2})}, \qquad (11)$$

$$p'(w) = -4H^{3/2}\,\frac{sn(2wH^{1/2})dn(2wH^{1/2})}{\left\{1-cn(2wH^{1/2})\right\}^2}, \qquad (11a)$$

where sn, cn and dn are the Jacobian elliptic functions,

$$H^2 = (e_2-e_1)(e_2-e_3) = 2e_2^2 + \frac{g_3}{3e_2}$$

and e_1, e_2, e_3 are the roots of the equation

$$4w^3 - g_2 w - g_3 = 0,$$

e_2 being the real root, whereas e_1 and e_3 are complex. In our case $e_2 = -1/\sqrt[3]{4} = -0.6300$.

The independent variable η is expressed by the integral of the p-function, i.e.

$$\eta = 6^{2/3} \int_w^{w_o} [e_2 + H \frac{1+cn(2wH^{1/2})}{1-cn(2wH^{1/2})}] dw \qquad (7b)$$

The result is in close agreement with the results obtained by simple quadrature of an equation corresponding to (7).

The velocity distribution in the direction parallel to the plane of the rods, but perpendicular to their axes is given by the equation of continuity

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \qquad (12)$$

where according to Eq. (5)

$$\frac{\partial u_1}{\partial x} = -2A_o x^{-2} F_1(y) \qquad (5a)$$

and therefore

$$v_1 = cA_o x^{-2} \int F_1(c\eta) d\eta \qquad (12a)$$

We now use the expressions of $F_1(c\eta)$ from Eq. (6b), $d\eta$ from Eq. (7) and $z = p(w)$ from Eq. (9), obtaining

$$v_1 = 6^{2/3} Acx^{-2} \int p(w) \; p'(w) dw$$
$$= (\tfrac{9}{2})^{1/3} Acx^{-2} [p^2(w) - p^2(w_o)]$$

Expressing the p-function by the Jacobian elliptic functions we

get a velocity distribution in close agreement with that found by simple quadrature (Fig. 3).

4. Summary. The velocity distribution behind a system of cylindrical equidistant rods is found by using the expression of L. Prandtl for the shear stress in turbulent motion (mixing length theory). The solution of the differential equation is given in terms of the elliptic functions of Weierstrass, which for the numerical calculation are transformed to the elliptic functions of Jacobi. The results are in close agreement with those obtained earlier by simple quadrature.

5. Bibliography.

[1] R. Gran Olsson, Zeitschr. f. angew. Math. u. Mech. vol. 16, (1936), p. 257-274, see also S. Goldstein, Modern Developments in Fluid Dynamics, Oxford University Press, London, 1938, vol. II, p. 586 and p. 667.

[2] L. Prandtl, Essentials of Fluid Dynamics, Blackie & Son Ltd., London and Glasgow 1952, p. 118.

[3] E. H. Neville, Jacobian Elliptic Functions, Oxford University Press, London, 2 ed. 1951, p. 26-49.

[4] L. M. Milne-Thomson, Jacobian Elliptic Function Tables, Dover Publications Inc., New York 1950, p. 23-24.

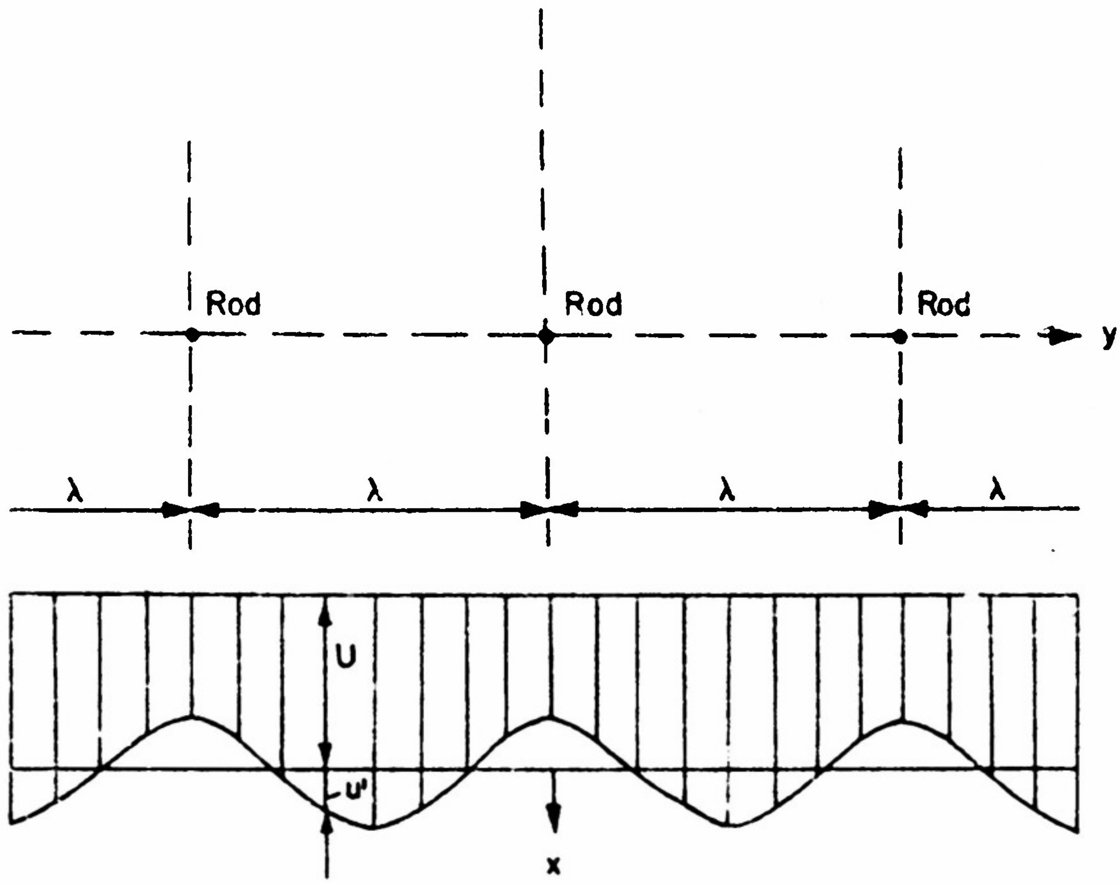

FIG. 1

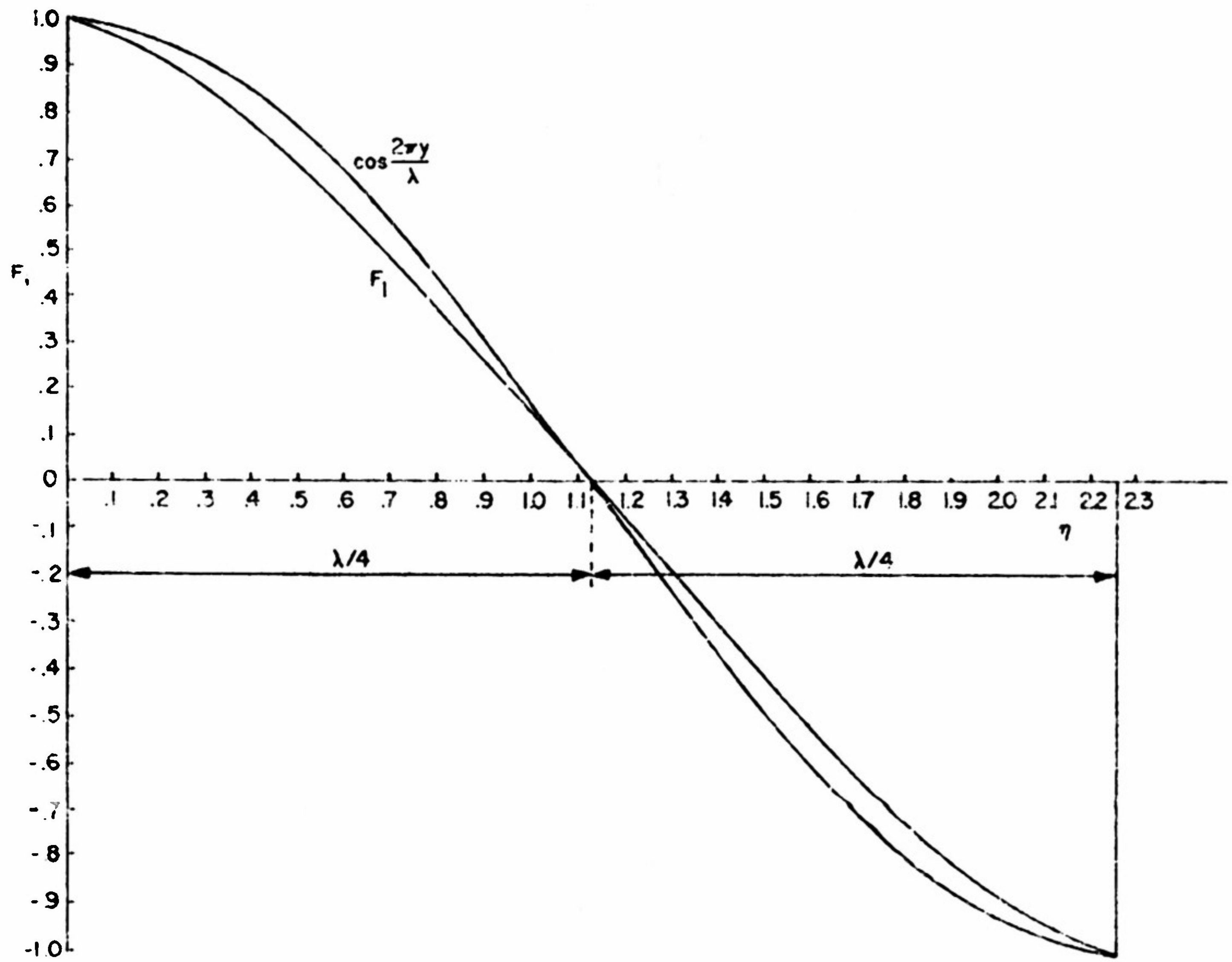
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
F_1
$\cos \frac{2\pi y}{\lambda}$
F_1
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2
2.3
η
$\lambda/4$
$\lambda/4$

FIG. 2

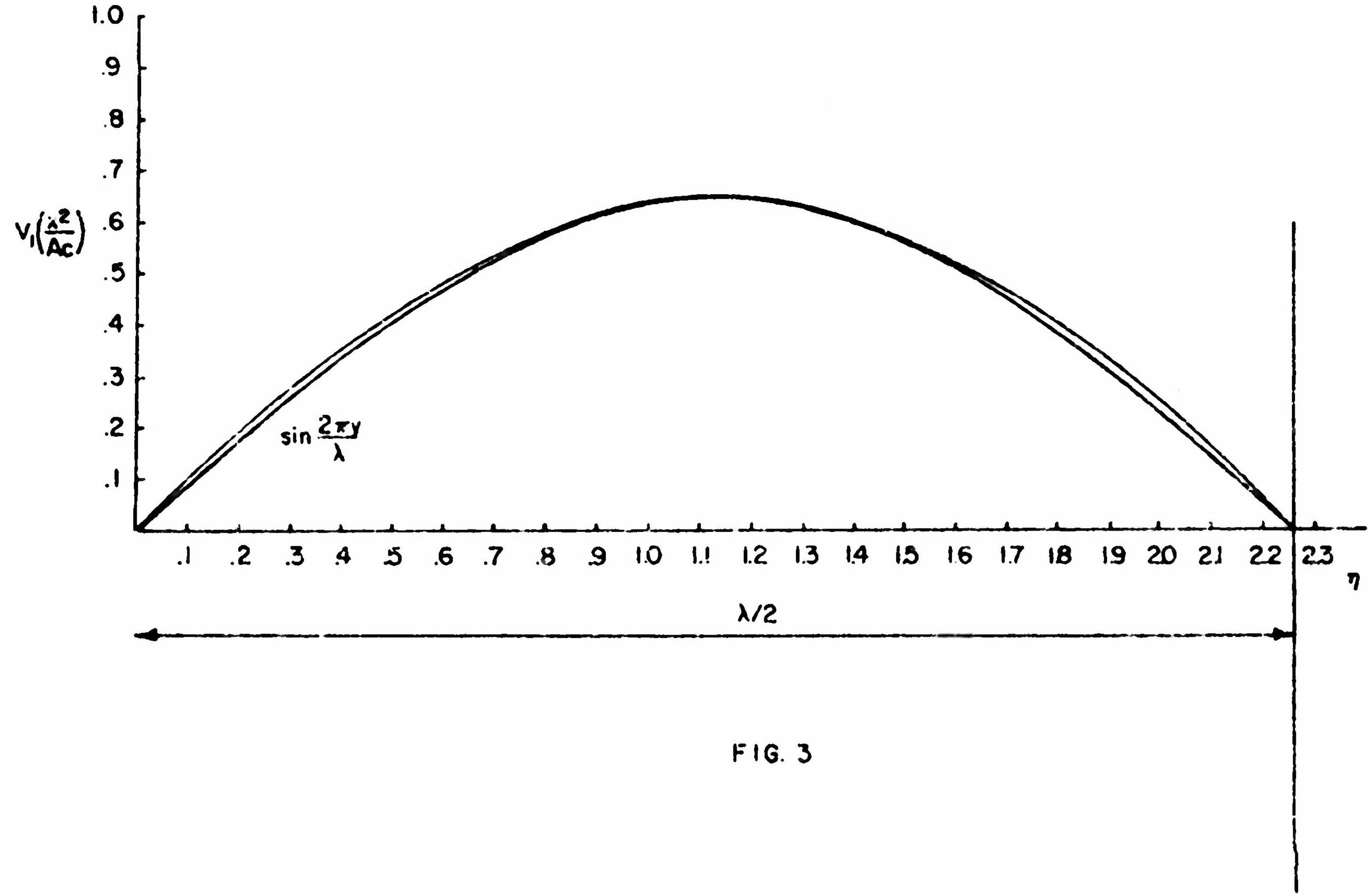
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
V1(λ2/Ac)
sin 2πy/λ
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0 2.1 2.2 2.3
η
λ/2

FIG. 3